The axisymmetric antidynamo theorem revisited

R. Kaiser*
Fakultät für Mathematik, Physik und Informatik
Universität Bayreuth
D-95440 Bayreuth, Germany

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Abstract

The axisymmetric kinematic dynamo problem is reconsidered and a number of open questions are answered. Apart from axisymmetry and smoothness of data and solution we deal with this problem under quite general conditions, i.e. we assume a compressible fluid of variable (in space and time) conductivity moving in an arbitrary (axisymmetric) domain. We prove unconditional, pointwise and exponential decay of magnetic field and electric current to zero. The decay rate of the external (meridional) magnetic field can become very small (compared to free decay) for special flow fields and large magnetic Reynolds numbers. We give an example of that. On the other hand, we show for incompressible fluids of constant conductivity in balls that the meridional and azimuthal decay rates coincide with the well-known poloidal and toroidal free decay rates, respectively.

Key Words: Magnetohydrodynamics, dynamo theory, axisymmetric theorem.

1 Introduction

The axisymmetric (or Cowling's) theorem is the oldest and most renowned of all antidynamo theorems which generally preclude under certain assumptions the maintenance of the magnetic field against ohmic loss by the dynamo process. Here, the assumption is axisymmetry (of magnetic field, flow field, conductivity distribution and shape of the conductor). This theorem had considerable significance in the early days of dynamo theory, where convincing analytical arguments, computational or even experimental evidence for the dynamo process were yet far away. In view of nearly axisymmetric magnetic fields of the earth and the sun this theorem called in question the dynamo process as such. No wonder that after its very first provisional formulation by Cowling (1934) researchers

*Email: ralf.kaiser@uni-bayreuth.de

in the field tried to improve the theorem with respect to greater mathematical rigor, stronger statements or weaker assumptions. The history of this endeavour has already been recorded in sufficient detail elsewhere. I mention only the excellent articles by Ivers & James (1984), who cover the situation up to 1983, by Fearn et al. (1988), who note in particular that this endeavour is not yet finished, and by Núñez (1996), whose review presents the state of the art and emphasizes especially the role of Lortz and his coworkers. So, I refrain from giving another detailed account of the history but distinguish roughly three phases in the development of the axisymmetric theorem.

In the first phase the *stationary* problem with solenoidal flow field and constant conductivity was in the focus. Cowling himself demonstrated the impossibility of a purely meridional magnetic field sustained by a purely meridional flow field by means of his now famous 'neutral point argument'. Backus & Chandrasekhar (1956) generalized this result by admitting meridional as well as azimuthal components for the magnetic field and the flow field, and they replaced the somewhat informal neutral point argument by mathematically more rigorous maximum principles for elliptic equations. Still later Lortz (1968) realized that these elliptic equations are incompatible with nontrivial solutions even for non-solenoidal flow fields and variable (in space) conductivity.

In the second phase the more realistic *time-dependent* problem came into focus but still under the proviso of a solenoidal flow field and constant conductivity. With these assumptions energy methods and variational inequalities can be applied to prove exponential decay in the energy norm of the meridional scalar and, in the absence of the meridional field, of the azimuthal field. Backus (1957) and Braginskii (1965) did so and gave, moreover, for balls explicit (albeit different) decay rate bounds on these quantities. Note that Backus (1957) considered already at that time the meridional problem without the solenoidal-flow and constant-conductivity asumptions and found exponential decay; however, in order to evaluate the variational problem, he had to introduce unsatisfactory ad-hoc assumptions.

The third phase was initiated by a paper of Todoeschuck & Rochester (1980), who, in the view of Saturn's almost axisymmetric magnetic field, argued that Cowling's theorem might have loopholes, in particular, that non-stationary axisymmetric magnetic fields could be generated by non-solenoidal flows in a compressible fluid. Several researchers, however, dismissed this possibility and tried to close the loopholes; so, the third phase was characterized by attempts to prove the axisymmetric theorem for *non-solenoidal flows* and *variable conductivity.* The first such attempt was due to Hide & and Palmer (1982), who demonstrated monotonous decay of the meridional scalar in the maximum norm. Their arguments were ingenious but, from a mathematical point of view, not rigorous (cf. Núñez 1996). Lortz & Meyer-Spasche (1982a) gave a rigorous proof of this result using a clever combination of parabolic and elliptic maximum principles. Ivers & James (1984) combining these maximum principles with skillfully chosen comparison functions strengthened this result in that they showed exponential decay to zero together with explicit decay rate bounds.

Less is known about the azimuthal field. First of all the azimuthal problem so far has

exclusively been considered in the case that the meridional field has already died out and does not provide a source term for the azimuthal field. With this assumption Lortz & Meyer-Spasche (1982b) proved the azimuthal field to be bounded in an integral norm for all times in terms of its initial value. Their method of proof depends of the existence of positive solutions of an auxiliary problem, which requires some mathematical effort. Ivers & James (1984) obtained the same result using some more intuitive arguments. Finally, using heavy mathematical machinery, Lortz, Meyer-Spasche & Stredulinsky (1984) proved in a *plane* model problem exponential decay of the azimuthal field to zero even in the maximum norm.

Despite all these achievements there are still a number of shortcomings that need to be removed before the axisymmetric theorem can be regarded as fully established:

(i) In an axisymmetric setting the evolution of the meridional field decouples from that of the azimuthal field (which is the essence of the axisymmetric theorem), but not vice versa. Nevertheless, the evolution of the azimuthal field so far has exclusively been considered without meridional source term. So, a proof of the axisymmetric theorem using the full (coupled) set of dynamo equations is still missing.

(ii) Meridional decay results refer so far to the meridional scalar, for which a tractable equation is at hand, and not to the meridional field itself, with two notable exceptions: Hide (1981) argued that a signed version of the total magnetic flux leaving/entering the conductor has to decay in the case of axisymmetry; his arguments, however, are not rigorous. Ivers & James (1984) appeal to Schauder estimates for elliptic and parabolic equations to relate the decay of the meridional field to that of the meridional scalar; these estimates, however, are not uniform in space; they fail, in particular, at the magnetic axis and at the conductor's boundary leaving thus open - however unlikely - the possibility of magnetic regeneration there.

(iii) Even in the absence of the meridional source term, unconditional decay to zero of the azimuthal field is not yet established. Ivers & James (1984) discuss a 'scenario for decay to zero' based on additional ad-hoc assumptions, whereas the result of Lortz et al. (1984), although highly suggestive, does not make a statement about the *axisymmetric* problem.

(iv) The only explicit decay rate bounds valid for general flow fields and conductivity distributions refer to the meridional field and are due to Ivers & James (1984). These bounds become extremely small already for moderate values of the magnetic Reynolds number R_m. For instance, considering the earth, one finds for $R_m = 10^2$ (which is a rough estimate) magnetic decay rate bounds such that the corresponding decay times greatly exceed the lifetime of the earth. So, the question arises how close these bounds are to actual decay rates.

(v) In the case of solenoidal flow and constant conductivity the optimal decay rates are those of axisymmetric free decay. In balls these can be compared to the well-known poloidal/toroidal free decay rates. Calculations by Backus (1957) suggest here a difference whereas Braginskii (1965) notes (without proof) no such difference.

We prove in the following unconditional, pointwise and exponential decay to zero of

the meridional and the azimuthal fields satisfying the coupled dynamo equations, thus settling, finally, problems (i) to (iii) of the above list. Problem (iv) is resolved by example. We prove for a special piecewise constant flow field that the meridional decay rate shrinks exponentially fast to zero with respect to R_m. This behaviour does not occur in simpler 'one-dimensional' flows. For purely radial or purely non-radial flows it is proved that meridional decay rates do not drop significantly below the corresponding free decay rates, irrespective of R_m. Concerning the last problem (v) we derive the governing equations of meridional and azimuthal free decay. In balls we find the corresponding decay rates equal to those of poloidal and toroidal free decay. We thus verify the assertions of Braginskii, but make also a comment on Backus' differing result.

In proving these results we make use of all the mathematical tools already applied by previous authors, in particular, we use variational methods as in (Backus 1957), comparison functions as in (Ivers & James 1984) and positive auxiliary solutions as in (Lortz et al. 1984). The main idea, however, which allows to go beyond these authors, is a consistent use of the 5-dimensional formulation of the axisymmetric problem. This formulation eliminates the coordinate singularity inherent to the 3-dimensional axisymmetric formulation and makes the problem accessible to more or less standard techniques for partial differential equations. For instance, Galerkin representations of solutions are essential in the present context. These are standard for the azimuthal field and have recently been provided for the meridional scalar (Kaiser & Uecker 2009). For solutions in this form we can use a 'higher-order-decay theorem' relating the decay of the solution to the decay of its higher derivatives (Kaiser 2012b). This result allows, in particular, to relate the decay of the meridional scalar to that of the meridional field, and, moreover, to relate integral decay to pointwise decay.

Concerning the organization of the material I have tried to make the paper accessible also to those readers who are less inclined to mathematics but are interested in the status of the axisymmetric theorem. So, some more formal aspects of the paper, some proofs and some auxiliary calculations have either been omitted (and published elsewhere) or shifted to appendices, and the trustful reader might concentrate on the conclusions in the main text. In more detail the paper is organized as follows: section 2 presents the formulation of the axisymmetric dynamo problem in 2, 3 and 5 dimensions, which emphasize different aspects of the problem. In section 3 the higher-order-decay theorem is presented and then applied to the meridional decay problem whereas in section 4 results by Lortz et al. (1984) are adjusted to the axisymmetric situation and then applied to the azimuthal decay problem. A more technical part of this section has been shifted to appendix A. Section 5 is devoted to the construction of meridional sub- and supersolutions to obtain explicit upper and lower decay rate bounds on the external magnetic field for certain flow fields and section 6, finally, presents sharp meridional and azimuthal decay rate bounds for incompressible fluids of constant conductivity. Technical details have again been shifted to the appendices B and C.

2 Formulation of the axisymmetric dynamo problem

In the framework of magnetohydrodynamics the kinematic dynamo problem is the following initial value problem in all space (Backus 1958, Moffatt 1978)

$$\left.\begin{array}{cl} \partial_t \mathbf{B} = -\nabla\times(\eta\nabla\times\mathbf{B}) + \nabla\times(\mathbf{v}\times\mathbf{B})\,,\ \nabla\cdot\mathbf{B}=0 & \text{in } G\times\mathbb{R}_+, \\ \nabla\times\mathbf{B}=0\,,\quad \nabla\cdot\mathbf{B}=0 & \text{in } \widehat{G}\times\mathbb{R}_+, \\ \mathbf{B} \text{ continuous} & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \mathbf{B}(\mathbf{x},\cdot)\to 0 & \text{for } |\mathbf{x}|\to\infty, \\ \mathbf{B}(\cdot\,,0)=\mathbf{B}_0\,,\quad \nabla\cdot\mathbf{B}_0=0 & \text{on } G\times\{t=0\}. \end{array}\right\} \tag{2.1}$$

Here, the induction equation $(2.1)_1$ describes the generation of the magnetic field $\mathbf{B}$ by the motion (with prescribed flow field $\mathbf{v}$) of a conducting fluid (with magnetic diffusivity $\eta>0$) in a bounded region $G\subset\mathbb{R}^3$. Outside the fluid region there are no further sources of magnetic field. Thus, $\mathbf{B}$ matches continuously to some vacuum field in $\widehat{G}:=\mathbb{R}^3\setminus\overline{G}$ that vanishes at spatial infinity.

The central simplifying assumption of the present paper is axisymmetry of all variables appearing in (2.1) including the shape ∂G of the conductor. Using cylindrical coordinates (ρ,ϕ,z) with $\mathbf{e}_z$ pointing in the direction of the symmetry axis S and (ρ,ϕ) being polar coordinates in the planes perpendicular to S, this assumption implies the following representation of the solenoidal field $\mathbf{B}$

$$\mathbf{B} = \nabla M\times\nabla\phi + A\nabla\phi = \Big(-\frac{1}{\rho}\,\partial_z M\,\mathbf{e}_\rho + \frac{1}{\rho}\,\partial_\rho M\,\mathbf{e}_z\Big) + \frac{1}{\rho}\,A\,\mathbf{e}_\phi = \mathbf{B}_m + \mathbf{B}_a. \tag{2.2}$$

The meridional field $\mathbf{B}_m$ is confined to planes $\phi = \mathit{const}$ and thus everywhere perpendicular to the azimuthal field $\mathbf{B}_a$. The corresponding scalars M and A depend on ρ, z and t but not on ϕ.

Inserting (2.2) into $(2.1)_{1a}$ and using the abbreviation $\mathbf{E}:=\eta\nabla\times\mathbf{B}-\mathbf{v}\times\mathbf{B}$ yields componentwise

$$\left.\begin{array}{l} \partial_z(\partial_t M + \rho E_\phi) = \partial_\phi E_z, \\ \partial_t A + \partial_z E_\rho - \partial_\rho E_z = 0, \\ \partial_\rho(\partial_t M + \rho E_\phi) = \partial_\phi E_\rho. \end{array}\right\} \tag{2.3}$$

The right-hand sides of $(2.3)_{1,3}$ vanish by assumption and one concludes

$$\partial_t M + \rho\big(\eta\nabla\times(\nabla M\times\nabla\phi + A\nabla\phi) - \mathbf{v}\times(\nabla M\times\nabla\phi + A\nabla\phi)\big)_\phi = c(t), \tag{2.4}$$

where $c(t)$ is an undetermined function of t. Evaluating the left-hand side of (2.4) and replacing $M-\int^t c(\tau)\,\mathrm{d}\tau$ by M, which does not affect the meridional field, we arrive at

$$\partial_t M - \eta\Delta_* M + v_\rho\,\partial_\rho M + v_z\,\partial_z M = 0, \tag{2.5}$$

where Δ_* denotes the elliptic operator $\partial_\rho^2 - 1/\rho\,\partial_\rho + \partial_z^2$.

On the other hand, evaluating $(2.3)_2$ yields an evolution equation for the azimuthal scalar A with a source term depending on M:

$$\partial_t A - \rho\,\partial_\rho\Big(\frac{\eta}{\rho}\,\partial_\rho A\Big) - \partial_z(\eta\,\partial_z A) + \rho\,\partial_\rho\Big(\frac{v_\rho}{\rho}\,A\Big) + \partial_z(v_z\,\partial_z A) = -\rho\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\partial_z M + \partial_z v_\phi\,\partial_\rho M. \tag{2.6}$$

Similarly but simpler, inserting (2.2) into $(2.1)_{2a}$ yields in the vacuum region

$$\Delta_* M = 0 \tag{2.7}$$

and $\partial_\rho A = \partial_z A = 0$, which implies by the asymptotic condition $(2.1)_4$,

$$A = 0. \tag{2.8}$$

The matching condition $(2.1)_3$ implies for M the matching condition

$$M,\ \partial_\rho M,\ \partial_z M \ \text{ continuous} \tag{2.9}$$

and for A by (2.8) the boundary condition

$$A\big|_{\partial G} = 0. \tag{2.10}$$

The asymptotic condition $(2.1)_4$ reads for M

$$\frac{1}{\rho}\,\partial_\rho M(\rho,z,\cdot) \to 0\,,\ \frac{1}{\rho}\,\partial_z M(\rho,z,\cdot) \to 0 \quad \text{for} \quad \rho^2 + z^2 \to \infty, \tag{2.11}$$

where this notation implies that the convergence is uniform on compact sets of the variables represented by dots. The coordinate singularity at $\rho = 0$ inherent to the representation (2.2) requires additional conditions on M and A to avoid spurious solutions:

$$M(\rho,\cdot,\cdot) \to 0\,,\ \partial_\rho M(\rho,\cdot,\cdot) = O(\rho)\,,\ \partial_z M(\rho,\cdot,\cdot) = O(\rho) \quad \text{for} \quad \rho \to 0 \tag{2.12}$$

and

$$A(\rho,\cdot,\cdot) = O(\rho^2) \quad \text{for} \quad \rho \to 0. \tag{2.13}$$

The conditions $(2.12)_{b,c}$ ensure a finite magnetic field on the symmetry axis S and imply, moreover, the finite limit $\lim_{\rho\to 0} M(\rho,\cdot,t) =: M_S(t)$. As $M_S(t)$ does not affect the meridional field we set it to zero. Condition (2.13) ensures a differentiable (with respect to ρ) magnetic field component B_ϕ that vanishes at S. Finally, $\mathbb{R}^3$ is replaced by the half-plane $H := \{(\rho,z) \in \mathbb{R}_+ \times \mathbb{R}\}$ and G and $\widehat{G}$ by the 'cross-sections' $G_2 := \{(\rho,z) \in H : \mathbf{x}(\rho,0,z) \in G\}$ and $\widehat{G}_2 := H \setminus \overline{G_2}$, respectively.

So, summarizing the foregoing results, a solution of the axisymmetric dynamo problem consists in a couple (M,A) with M satisfying (2.5) in $G_2 \times \mathbb{R}_+$ with initial value M_0, (2.7) in $\widehat{G}_2 \times \mathbb{R}_+$ and (2.9), (2.11) and (2.12) in $H \times \mathbb{R}_+$ and, furthermore, with A satisfying

(2.6) and (2.10) in $G_2 \times \mathbb{R}_+$ with initial value A_0. The initial values themselves have to satisfy (2.12) and (2.13) and typically some ‘compatibility conditions’ (cf. Kaiser 2012a).

This formulation in terms of the potentials M and A and the spatial variables ρ and z may be called minimal since it completely resolves the condition of axisymmetry. Most important, it yields an evolution equation for M decoupled from A and without zeroth-order term. However, there are also drawbacks. The divergence character of the azimuthal equation is veiled and, more important, we have to deal with singular coefficients in both equations. In fact, more redundant formulations turn out to be useful using cartesian coordinates in $\mathbb{R}^3$ and even $\mathbb{R}^5$ and using modified potentials.

The 3-dimensional formulation is straightforward. With $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and the abbreviation $\rho := (x^2 + y^2)^{1/2}$ the meridional problem can be summarized as follows:[1]

$$\left.\begin{array}{lr}
\partial_t M - \eta\Delta M + 2\eta\,\dfrac{\nabla\rho}{\rho}\cdot\nabla M + \mathbf{v}\cdot\nabla M = 0 & \text{in } G\times\mathbb{R}_+,\\
\Delta M - 2\,\dfrac{\nabla\rho}{\rho}\cdot\nabla M = 0 & \text{in } \widehat{G}\times\mathbb{R}_+,\\
M \text{ and } \nabla M \text{ continuous} & \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
M(x,y,\cdot,\cdot)\to 0\,,\ \nabla M(x,y,\cdot,\cdot) = O(\rho) & \text{for } \rho\to 0,\\
\dfrac{1}{\rho}\nabla M(\mathbf{x},\cdot)\to 0 & \text{for } |\mathbf{x}|\to\infty,\\
M(\cdot\,,0) = M_0 & \text{on } G\times\{t=0\}.
\end{array}\right\} \qquad (2.14)$$

Here, ∇ denotes the Cartesian gradient vector $(\partial_x, \partial_y, \partial_z)$ and in $(2.14)_{1,2}$ we made use of the identity for axisymmetric functions,

$$\Delta_* M = \partial_\rho^2 M - \frac{1}{\rho}\,\partial_\rho M + \partial_z^2 M = \Delta M - 2\,\frac{\nabla\rho}{\rho}\cdot\nabla M.$$

Concerning the azimuthal problem we introduce the variable $\mathcal{A} := A/\rho^2$ and use the identity for axisymmetric functions

$$\frac{1}{\rho}\,\partial_\rho\Big(\frac{\eta}{\rho}\,\partial_\rho\big(\rho^2\mathcal{A}\big)\Big) + \partial_z\big(\eta\,\partial_z\mathcal{A}\big) = \nabla\cdot\Big(\frac{\eta}{\rho^2}\nabla\big(\rho^2\mathcal{A}\big)\Big)$$

to obtain

$$\left.\begin{array}{lr}
\partial_t\mathcal{A} - \nabla\cdot\Big(\dfrac{\eta}{\rho^2}\nabla\big(\rho^2\mathcal{A}\big)\Big) + \nabla\cdot(\mathbf{v}\mathcal{A}) = \nabla\cdot\Big(\mathbf{B}_m\,\dfrac{v_\phi}{\rho}\Big) & \text{in } G\times\mathbb{R}_+,\\
\mathcal{A} = 0 & \text{on } \partial G\times\mathbb{R}_+,\\
\mathcal{A}(\cdot\,,0) = \mathcal{A}_0 & \text{on } G\times\{t=0\}.
\end{array}\right\} \qquad (2.15)$$

Note that a solution of (2.15) that is well-defined on the symmetry axis automatically satisfies (2.13).

[1] To keep the notation simple we do not use different symbols for the same function depending on different coordinates. The arguments should be clear from the context.

The 3-dimensional formulation of the azimuthal problem makes the divergence character manifest; the coefficients, however, are still singular. This problem can be cured by a five-dimensional formulation. Let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, ∇ the corresponding cartesian gradient and Δ the 5-dimensional Laplacian. Identifying ρ^2 with $\sum_{i=1}^4 x_i^2$ and z with x_5 and defining $G_5 := \{\mathbf{x} \in \mathbb{R}^5 : (\rho, z) \in G_2\}$, axisymmetric functions in $\mathbb{R}^3$ can be considered as axisymmetric functions in $\mathbb{R}^5$. Introducing the variable $\mathcal{M} := M/\rho^2$ and the 5-dimensional meridional flow field $\mathbf{v}_m^{(5)} := \sum_{i=1}^4 (v_\rho/\rho) x_i \mathbf{e}_i + v_z \mathbf{e}_5$, and observing the axisymmetric identity

$$\Delta_* M = \frac{1}{\rho}\,\partial_\rho\Big(\rho^3 \partial_\rho\Big(\frac{1}{\rho^2}\,M\Big)\Big) + \partial_z^2 M = \rho^2 \nabla\cdot\nabla\Big(\frac{1}{\rho^2}\,M\Big) = \rho^2 \Delta\mathcal{M},$$

the meridional problem takes the form

$$\left.\begin{array}{ll}
\partial_t \mathcal{M} - \eta\Delta\mathcal{M} + \mathbf{v}_m^{(5)}\cdot\nabla\mathcal{M} + 2\,\dfrac{v_\rho}{\rho}\,\mathcal{M} = 0 & \text{in } G_5 \times \mathbb{R}_+, \\
\Delta\mathcal{M} = 0 & \text{in } \widehat{G}_5 \times \mathbb{R}_+, \\
\mathcal{M} \text{ and } \nabla\mathcal{M} \text{ continuous} & \text{in } \mathbb{R}^5 \times \mathbb{R}_+, \\
\mathcal{M}(\mathbf{x},\cdot) \to 0 & \text{for } |\mathbf{x}| \to \infty, \\
\mathcal{M}(\cdot\,,0) = \mathcal{M}_0 & \text{on } G_5 \times \{t=0\}.
\end{array}\right\} \tag{2.16}$$

Note that for a smooth axisymmetric flow field v_ρ/ρ is well-defined on S; thus, no singular coefficients appear any more in (2.16). No axis-condition on $\mathcal{M}$ is anymore necessary; condition (2.12) is automatically satisfied for any well-defined solution of (2.16). Moreover, outside the conductor, $\mathcal{M}$ is a harmonic potential (in 5 dimensions); using this information (cf. Kaiser & Uecker 2009, Appendix C) condition $(2.16)_4$ implies, in particular,

$$\nabla\mathcal{M}(\mathbf{x},\cdot) = O(|\mathbf{x}|^{-4}) \quad \text{for } |\mathbf{x}| \to \infty,$$

which again implies (2.11). Note, however, that for these advantages we had to pay a prize, viz. a zeroth-order term in $(2.16)_1$.

Concerning the azimuthal problem we make use of the axisymmetric identities

$$\rho\,\partial_\rho\Big(\frac{\eta}{\rho}\,\partial_\rho A\Big) + \partial_z(\eta\,\partial_z A) = \rho^2\,\nabla\cdot\Big(\eta\nabla\frac{A}{\rho^2}\Big) + 2\,\frac{\partial_\rho\eta}{\rho}\,A$$

and

$$\rho\,\partial_\rho\Big(\frac{v_\rho}{\rho}\,A\Big) + \partial_z v_z\,A = \rho^2\,\nabla\cdot\Big(\mathbf{v}_m^{(5)}\,\frac{A}{\rho^2}\Big) - 2\,\frac{v_\rho}{\rho}\,A$$

to obtain

$$\left.\begin{array}{ll}
\partial_t\mathcal{A} - \nabla\cdot(\eta\nabla\mathcal{A}) + \nabla\cdot(\mathbf{v}_m^{(5)}\mathcal{A}) - 2\Big(\dfrac{\partial_\rho\eta}{\rho} + \dfrac{v_\rho}{\rho}\Big)\mathcal{A} & \\
\qquad = \mathrm{d}\mathbf{v}_a^{(5)}\cdot\nabla\mathcal{M} + 2\,\dfrac{\partial_z v_\phi}{\rho}\,\mathcal{M} & \text{in } G_5 \times \mathbb{R}_+, \\
\mathcal{A} = 0 & \text{on } \partial G_5 \times \mathbb{R}_+, \\
\mathcal{A}(\cdot\,,0) = \mathcal{A}_0 & \text{on } G_5 \times \{t=0\},
\end{array}\right\} \tag{2.17}$$

where we used the abbreviation $\mathrm{d}\mathbf{v}_a^{(5)} := \sum_{i=0}^{4} \partial_z v_\phi\,(x_i/\rho)\mathbf{e}_i - \rho\,\partial_\rho(v_\phi/\rho)\mathbf{e}_5$. On the premises of smooth (up to second order) axisymmetric data η and $\mathbf{v}$, all coefficients in (2.17) are now well-defined on S (cf. Ivers & James 1984). But, again, we have bartered for this advantage a zeroth-order term that destroys the divergence character of the left-hand side in $(2.17)_1$.

As far as the various forms of meridional and azimuthal evolution equations have been considered by previous authors, these coincide, of course, with ours. Concerning the asymptotic conditions at S and at infinity, however, there are some minor differences. For example, Ivers & James (1984) use the condition

$$M(\rho, z, \cdot) = O\big((\rho^2 + z^2)^{-1/2}\big) \quad \text{for} \quad \rho^2 + z^2 \to \infty \tag{2.18}$$

instead of (2.11). Stredulinski et al. (1986) called condition (2.18) in question. The discrepancy, however, seems to be due to a confusion about the axisymmetric and the plane-symmetric problem (cf. Kaiser & Uecker 2009, Remark B.1). In fact, in view of the five-dimensional formulation conditions (2.11) and (2.18) are equivalent for solutions of (2.7). A similar remark holds for the axis-condition (2.12), which is typically replaced by previous authors (as far as they specify at all an axis-condition) by the slightly weaker condition

$$M(\rho, \cdot, \cdot) = O(\rho^2) \quad \text{for} \quad \rho \to 0. \tag{2.19}$$

In fact, (2.19) is enough to establish the equivalence between the two- and the five-dimensional problem and hence the equivalence between (2.12) and (2.19) for meridional solutions.

Concerning the flow field some authors require the normal component to vanish on ∂G, others do not. In fact, some results depend on such a condition, others do not. In the following we will specify this condition where needed.

A final remark concerns the question, to which quantities axisymmetry has exactly to apply. In view of eqs. (2.3) Lortz (1968) required axisymmetry of the electrical quantities $\mathbf{B}$ and $\mathbf{E}$ but not of $\mathbf{v}$ or η. Todoeschuck & Rochester (1980) critized this approach as a disguise of the usual requirements. Lortz's assumptions are indeed enough to derive eqs. (2.5) and (2.6) even for non-axisymmetric coefficients $\mathbf{v}$ and η. However, evaluating the conditions $\partial_\phi E_\rho = \partial_\phi E_z = 0$ on their part yield the additional equations

$$\left.\begin{aligned} \partial_\phi \eta\, \partial_\rho A - \partial_\phi v_\rho\, A - \partial_\phi v_\phi\, \partial_z M = 0, \\ \partial_\phi \eta\, \partial_z A - \partial_\phi v_z\, A + \partial_\phi v_\phi\, \partial_\rho M = 0. \end{aligned}\right\} \tag{2.20}$$

These equations were trivially solved by axisymmetric coefficients (and this solution Todoeschuck & Rochester probably had in mind). Otherwise, for prescribed non-axisymmetric coefficients, (2.20) represent additional constraints. The combined system (2.5), (2.6) and (2.20) for axisymmetric scalars M and A, however, does not seem to be well-posed; in particular, general solvability cannot be expected. So, the present treatment of the axisymmetric problem cannot dispense with the assumption of axisymmetric coefficients.

3 Decay of meridional field and azimuthal current

In this section the exponential decay of the meridional scalar M in the maximum norm is related to exponential decay with (almost) the same decay rate of the meridional field $\mathbf{B}_m$ and of the azimuthal current $\mathbf{J}_a$. As discussed in the introduction pointwise decay of M, for instance in the form

$$|M(\rho,\cdot,t)| \le C g(\rho)\, e^{-d_0 t}, \qquad \rho > 0\,,\ t \ge 0 \tag{3.1}$$

with some $C > 0$, $d_0 > 0$ and a function $g(\rho) > 0$ with (at least) $g(\rho) = O(\rho)$ for $\rho \to 0$, is well established (see Ivers & James 1984) and is the prerequisite of the present section. We come back to this subject when discussing decay rates in section 5.

The central tool of this section is the following higher-order-decay theorem for smooth solutions u of the system

$$\left.\begin{array}{cl}
\partial_t u - a\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f & \text{in } G\times\mathbb{R}_+,\\
\Delta u = 0 & \text{in } \widehat{G}\times\mathbb{R}_+,\\
u \text{ and } \nabla u \text{ continuous} & \text{in } \mathbb{R}^n\times\mathbb{R}_+,\\
u(\mathbf{x},\cdot)\to 0 & \text{for } |\mathbf{x}|\to\infty,\\
u(\cdot\,,0) = u_0 & \text{on } G\times\{t=0\}.
\end{array}\right\} \tag{3.2}$$

Here, G is a bounded domain in $\mathbb{R}^n$ ($n \ge 3$) with complement $\widehat{G} = \mathbb{R}^n\backslash\overline{G}$ and (sufficiently smooth) boundary ∂G. Associated to (3.2) is the eigenvalue problem

$$\left.\begin{array}{cl}
-\Delta u = \lambda u & \text{in } G,\\
\Delta u = 0 & \text{in } \widehat{G},\\
u \text{ and } \nabla u \text{ continuous} & \text{in } \mathbb{R}^n,\\
u(\mathbf{x})\to 0 & \text{for } |\mathbf{x}|\to\infty,
\end{array}\right\} \tag{3.3}$$

which is well-defined and has a lowest eigenvalue λ_1 that is positive (see appendix C).

To measure the spatial smoothness of the coefficients a, $\mathbf{b}$ and c we use the notation $a \in C^k(\overline{G}\times\mathbb{R}_+)$ ($k \ge 0$), which means that all spatial derivatives up to order k are continuous and satisfy bounds of the form[2] $|D^k a| := \max_{|\alpha|\le k}\sup_{\overline{G}\times\mathbb{R}_+}|D^\alpha a| < K$ for some $K > 0$. $a \in C^k_1(\overline{G}\times\mathbb{R}_+)$ means that, additionally, $\partial_t u$ is a continuous function with corresponding bound.

Theorem 1 (Higher-order-decay) *Let a, $\mathbf{b}$, $c \in C^k(\overline{G}\times\mathbb{R}_+)$ ($k \ge 0$) with common bound $|D^k a|$, $|D^k\mathbf{b}|$, $|D^k c| < K$ and $a \ge a_0 > 0$. Let, furthermore, f satisfy the higher-order energy decay condition*

$$\sum_{|\alpha|\le k}\int_G |D^\alpha f(\mathbf{x},t)|^2 \mathrm{d}x \le C_f\, e^{-2d_f t}, \qquad t \ge 0 \tag{3.4}$$

[2] Here, α enumerates all derivatives D^α of order $\le k$.

with some constants $C_f > 0$ and $d_f > 0$, and let u be a classical (i.e. $u \in C^2_1(G \times \mathbb{R}_+)$) solution of (3.2). Then, the energy decay condition

$$\int_G |u(\mathbf{x},t)|^2 \mathrm{d}x \leq C_0\, e^{-2d_0 t}, \qquad t \geq 0 \tag{3.5}$$

implies the higher-order energy decay

$$\sum_{|\alpha| \leq k+1} \int_G |D^\alpha u(\mathbf{x},t)|^2 \mathrm{d}x \leq C\, e^{-2dt}, \qquad t \geq 0 \tag{3.6}$$

as well as decay in the maximum norm:

$$\sum_{|\alpha|=l} \sup_{\overline{G}} |D^\alpha u(\cdot,t)| \leq \widetilde{C}\, e^{-dt}, \qquad t \geq 0, \tag{3.7}$$

where $l < k+1-n/2$. Here, $d := \min\{d_i - \epsilon, d_0, d_f\}$, where $d_i := a_0\lambda_1$ is the 'intrinsic' decay rate of (3.2), and C and $\widetilde{C}$ are constants depending on G, the initial value u_0, K, C_0, C_f,n, k and $\epsilon > 0$.

The proof of this rather technical result can be found in (Kaiser 2012b). Yet some remarks are in order: the existence of classical solutions of the required type depends on the regularity of the coefficients (as specified in the theorem), on the regularity of the initial value u_0 and on suitable 'compatibility conditions' (for details see (Kaiser & Uecker 2009)). So, as a rule, for sufficiently regular data, conditions (3.4) and (3.5) imply decay to arbitrarily large orders. The decay rate of u is bounded by the lowest of the three decay rates d_i, d_0 and d_f; the intrinsic one d_i, however, can only 'almost' be attained since typically C and $\widetilde{C}$ diverge (at most algebraically) in the limit $\epsilon \to 0$.

The 5-dimensional formulation (2.16) of the meridional problem is obviously of the type to which Theorem 1 can be applied. So, setting $f = 0$, $n = 5$ and $k = 4$ the remaining condition (3.5) takes the form

$$\int_{G_5} \mathcal{M}(\mathbf{x},t)^2 \mathrm{d}x = |S_3| \int_{G_2} \left(\frac{1}{\rho^2} M(\rho,z,t)\right)^2 \rho^3 \mathrm{d}\rho\, \mathrm{d}z \leq C_0\, e^{-2d_0 t}, \tag{3.8}$$

where $|S_3|$ is the volume of the 3-dimensional sphere S_3. Condition (3.8) is clearly implied by (3.1) when observing the axis-condition of the radial function $g(\rho)$.

As consequences we have then (3.6):

$$\sum_{|\alpha| \leq 5} \int_{G_5} |D^\alpha \mathcal{M}(\mathbf{x},t)|^2 \mathrm{d}x \leq C\, e^{-2dt}, \qquad t \geq 0 \tag{3.9}$$

and (3.7) with $l = 0, 1$ and 2:

$$\sup_{\overline{G}_5} |\mathcal{M}(\cdot,t)| + \sum_{i=1}^{5} \sup_{\overline{G}_5} |\partial_{x_i} \mathcal{M}(\cdot,t)| \leq \widetilde{C}\, e^{-dt}, \qquad t \geq 0, \tag{3.10}$$

$$\sum_{i,j=1}^{5} \sup_{\overline{G}_5} |\partial_{x_i}\partial_{x_j}\mathcal{M}(\cdot,t)| \le \widetilde{C}\, e^{-dt}, \qquad t \ge 0, \tag{3.11}$$

which yield uniform pointwise bounds on

$$|\mathbf{B}_m| = \Big| -\frac{1}{\rho}\,\partial_z M \mathbf{e}_\rho + \frac{1}{\rho}\,\partial_\rho M \mathbf{e}_z \Big| = \frac{1}{\rho}\,|\nabla(\rho^2\mathcal{M})| = |2\,\mathcal{M}\nabla\rho + \rho\nabla\mathcal{M}|$$

and

$$|\mathbf{J}_a| = |\nabla\times\mathbf{B}_m| = \frac{1}{\rho}\,|\Delta_* M| = \rho|\Delta\mathcal{M}|,$$

respectively. As $\mathcal{M}$ is harmonic in $\widehat{G}_5$, continuous in $\mathbb{R}^5$ and vanishes at infinity, $\mathcal{M}$ takes its maximum and minimum in $\overline{G}_5$. This conclusion holds likewise for each component of $|\mathbf{x}|\nabla\mathcal{M}$. In fact, $|\mathbf{x}|\partial_{x_i}\mathcal{M}$ solves the elliptic equation with negative zeroth order term,

$$\Big(\Delta - 2\,\frac{\mathbf{x}}{|\mathbf{x}|^2}\cdot\nabla - \frac{2}{|\mathbf{x}|^2}\Big)u = 0 \quad \text{ in } \widehat{G}_5,$$

and, thus, obeys a maximum principle, too. With (3.10) we can, therefore, estimate for $t \ge 0$:

$$\begin{aligned}
\sup_H |\mathbf{B}_m(\cdot,\cdot,t)| \ &\le 2\sup_{\mathbb{R}^5}|\mathcal{M}(\cdot,t)| + \sum_{i=1}^{5}\sup_{\mathbb{R}^5}|\mathbf{x}||\partial_{x_i}\mathcal{M}(\cdot,t)| \\
&\le 2\sup_{\overline{G}_5}|\mathcal{M}(\cdot,t)| + \sum_{i=1}^{5}\sup_{\overline{G}_5}|\mathbf{x}||\partial_{x_i}\mathcal{M}(\cdot,t)| \\
&\le \max\{2,R\}\,\widetilde{C}\,e^{-dt} \ =: \widetilde{C}_{\mathbf{B}_m}\,e^{-d_m t},
\end{aligned} \tag{3.12}$$

where R is the radius of a ball enclosing the conductor. As $\mathbf{J}_a$ is restricted to the conductor one obtains directly by (3.11):

$$\sup_{\overline{G}_2}\Big|\frac{1}{\rho}\,\mathbf{J}_a(\cdot,\cdot,t)\Big| \le \sup_{\overline{G}_5}|\Delta\mathcal{M}(\cdot,t)| \le \widetilde{C}_{\mathbf{J}_a}\,e^{-d_m t}, \qquad t \ge 0. \tag{3.13}$$

According to Theorem 1 the decay rate bound d_m is given by $d_m = \min\{d_i - \epsilon, d_0\}$. In section 6 we identify the intrinsic decay rate d_i with the rate of free decay of the meridional scalar; thus, $d_0 \le d_i$. Comparing the decay of meridional field and meridional scalar we find, therefore, in the case $d_0 < d_i$ the same bound for both quantities and in the case $d_0 = d_i$ almost the same bound. Concerning the constants $\widetilde{C}_{\mathbf{B}_m}$ and $\widetilde{C}_{\mathbf{J}_a}$ we made no effort to determine the exact dependence on the various parameters listed in Theorem 1. So, numerical values could be very large. From an observational point of view, however, these can always be compensated by sufficiently large (compared to the decay time $1/d_m$) periods of time. Finally, the high degree of regularity of the data (boundedness of fourth-order derivatives) required by Theorem 1 is due to the method of proof, in particular, the use of embedding results in $\mathbb{R}^5$ neglecting the axisymmetry of the problem. Using more refined proof techniques the regularity requirements are expected to decrease.

4 Decay of azimuthal field and meridional current

In this section the unconditional decay of the azimuthal field $\mathbf{B}_a$ and of the meridional current $\mathbf{J}_m$ is proved exploiting the divergence-character of the governing equation in 3 dimensions as well as the nonsingular formulation in 5 dimensions. A central tool is a positive axisymmetric solution P of the following auxiliary problem

$$\left.\begin{aligned} \partial_t P - \nabla\cdot\Big(\frac{\eta}{\rho^2}\nabla\big(\rho^2 P\big)\Big) + \nabla\cdot(\mathbf{v}P) = 0 &\quad \text{in } G\times\mathrm{I\!R}_+,\\ \mathbf{n}\cdot\nabla P = 0 &\quad \text{on } \partial G\times\mathrm{I\!R}_+,\\ P(\cdot\,,0) = P_0 > 0 &\quad \text{on } G\times\{t=0\}, \end{aligned}\right\} \tag{4.1}$$

that differs from (2.15) in a vanishing right-hand side in $(4.1)_1$ and a Neumann boundary condition instead of a Dirichlet condition. All data G, η, $\mathbf{v}$ and P_0 are assumed to be axisymmetric, which implies that the solution is axisymmetric, too. The regularity requirements that ensure the existence of solutions are moderate; they are more detailed for the non-singular 5-dimensional formulation in appendix A. We mention here only that $G \in \mathrm{I\!R}^3$ is a bounded domain with smooth boundary ∂G and exterior normal $\mathbf{n}$, $\eta \in C^1(\overline{G}\times\mathrm{I\!R}_+)$ and $\mathbf{v}\in C(\overline{G}\times\mathrm{I\!R}_+)$ with $\mathbf{n}\cdot\mathbf{v}|_{\partial G} = 0$. Moreover, we need the bounds $|\mathbf{v}| \le K$, $|\nabla\rho\cdot\mathbf{v}/\rho| \le K$, $|\nabla\rho\cdot\nabla\eta/\rho| \le K$ for some $K>0$ and $\eta \ge \eta_0 > 0$ on $\overline{G}\times\mathrm{I\!R}_+$.

Theorem 2 (Positive auxiliary solution) *Let P be a solution of the axisymmetric problem (4.1) with initial value P_0 and let $\underline{P}_0$, $\overline{P}_0$ be some positive numbers that bound P_0:*

$$\underline{P}_0 \le P_0 \le \overline{P}_0 \qquad \text{in } G. \tag{4.2}$$

Then, positive bounds $\underline{P}$ and $\overline{P}$ exist such that

$$\underline{P} \le P(\cdot,t) \le \overline{P} \qquad \text{for } t\ge 0, \tag{4.3}$$

where $\underline{P}$ and $\overline{P}$ depend only on G, η_0, K and $\underline{P}_0$ and $\overline{P}_0$.

The proof of Theorem 2 consists in an adaptation to our situation of a powerful result by Lortz et al. (1984). The necessary changes are rather technical and are, therefore, deferred to appendix A.

We compute next the time-derivative of the quantity $\int_G \mathcal{A}^2/P\,\mathrm{d}x$. To avoid singular coefficients we consider first a 'regularized' domain $G_\epsilon := \{\mathbf{x}\in G : \rho(\mathbf{x}) > \epsilon\}$ $(\epsilon > 0)$ bounded by the surface $S_\epsilon := \{\mathbf{x}\in\partial G : \rho(\mathbf{x}) > \epsilon\}$ and the cylinder $C_\epsilon := \{\mathbf{x}\in G : \rho(\mathbf{x}) = \epsilon\}$ such that $\partial G_\epsilon = S_\epsilon \cup \overline{C}_\epsilon$. In the limit $\epsilon\to 0$, C_ϵ shrinks to the line segment $L := \{\mathbf{x}\in G : \rho(\mathbf{x}) = 0\}$. By $(2.15)_1$, $(4.1)_1$ and with the abbreviation $f := \mathbf{B}_m\cdot\nabla(v_\phi/\rho)$ one obtains[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G_\epsilon}\frac{\mathcal{A}^2}{P}\,\mathrm{d}x = \int_{G_\epsilon}\Big(2\,\frac{\mathcal{A}}{P}\,\partial_t\mathcal{A} - \frac{\mathcal{A}^2}{P^2}\,\partial_t P\Big)\mathrm{d}x$$

$$= 2\int_{G_\epsilon}\frac{\mathcal{A}}{P}\,f\,\mathrm{d}x + \int_{G_\epsilon}\Big\{2\,\frac{\mathcal{A}}{P}\,\nabla\cdot\Big(\frac{\eta}{\rho^2}\nabla\big(\rho^2\mathcal{A}\big) - \mathbf{v}\mathcal{A}\Big) - \frac{\mathcal{A}^2}{P^2}\,\nabla\cdot\Big(\frac{\eta}{\rho^2}\nabla\big(\rho^2 P\big) - \mathbf{v}P\Big)\Big\}\mathrm{d}x.$$

[3]For simplicity, dependence on t is suppressed in the subsequent calculations.

Integrating by parts in the last integral and using the boundary condition $(2.15)_2$ on S_ϵ, the velocity field $\mathbf{v}$ drops from the integral and we are left with

$$
\begin{aligned}
&-\int_{G_\epsilon}\Big\{2\,\nabla\Big(\frac{\mathcal{A}}{P}\Big)\cdot\frac{\eta}{\rho^2}\nabla(\rho^2\mathcal{A})-2\,\frac{\mathcal{A}}{P}\,\nabla\Big(\frac{\mathcal{A}}{P}\Big)\cdot\frac{\eta}{\rho^2}\nabla(\rho^2 P)\Big\}\mathrm{d}x\\
&-\int_{C_\epsilon}\Big\{2\,\frac{\mathcal{A}}{P}\,\frac{\eta}{\rho^2}\,\partial_\rho(\rho^2\mathcal{A})-\frac{\mathcal{A}^2}{P^2}\,\frac{\eta}{\rho^2}\,\partial_\rho(\rho^2 P)\Big\}\mathrm{d}s\\
&=-2\int_{G_\epsilon}\eta P\Big|\nabla\Big(\frac{\mathcal{A}}{P}\Big)\Big|^2\mathrm{d}x-\int_{C_\epsilon}\eta\,\Big(\frac{\mathcal{A}}{P}\,\partial_\rho\mathcal{A}-\frac{\mathcal{A}^2}{P^2}\,\partial_\rho P\Big)\mathrm{d}s-2\int_{C_\epsilon}\frac{\eta}{\rho}\,\frac{\mathcal{A}^2}{P}\,\mathrm{d}s.
\end{aligned}
$$

In the limit $\epsilon\to 0$ only the second surface integral survives with the result $-4\pi\int_L \eta\,\mathcal{A}^2/P\,\mathrm{d}z$. In summary we have in the limit $\epsilon\to 0$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_G\frac{\mathcal{A}^2}{P}\,\mathrm{d}x=-2\int_G\eta P\Big|\nabla\Big(\frac{\mathcal{A}}{P}\Big)\Big|^2\mathrm{d}x-4\pi\int_L\eta\,\frac{\mathcal{A}^2}{P}\,\mathrm{d}z+2\int_G\frac{\mathcal{A}}{P}\,f\,\mathrm{d}x. \tag{4.4}
$$

By means of the bounds (4.3) on P, the variational inequality

$$
\int_G|\nabla g|^2\,\mathrm{d}x+2\pi\int_L|g|^2\,\mathrm{d}z\geq\mu_1\int_G|g|^2\,\mathrm{d}x
$$

for axisymmetric differentiable functions g vanishing on ∂G (see section 6), the bound (3.12) on $\mathbf{B}_m$ and the bound $|\nabla(v_\phi/\rho)|\leq K$, (4.4) can be estimated as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_G\frac{\mathcal{A}^2}{P}\,\mathrm{d}x\ &\leq -2\eta_0\,\underline{P}\Big\{\int_G\Big|\nabla\Big(\frac{\mathcal{A}}{P}\Big)\Big|^2\mathrm{d}x+2\pi\int_L\frac{\mathcal{A}^2}{P^2}\,\mathrm{d}z\Big\}+2\int_G\frac{\mathcal{A}}{P}\,\mathbf{B}_m\cdot\nabla\Big(\frac{v_\phi}{\rho}\Big)\mathrm{d}x\\
&\leq -2\eta_0\mu_1\,\underline{P}\int_G\frac{\mathcal{A}^2}{P^2}\,\mathrm{d}x+2KC_{\mathbf{B}_m}\,e^{-d_m t}\,\frac{1}{\underline{P}^{1/2}}\int_G\frac{\mathcal{A}}{P^{1/2}}\,\mathrm{d}x\\
&\leq -2\eta_0\mu_1\frac{\underline{P}}{\overline{P}}\int_G\frac{\mathcal{A}^2}{P}\,\mathrm{d}x+2KC_{\mathbf{B}_m}\,e^{-d_m t}\,\frac{|G|^{1/2}}{\underline{P}^{1/2}}\Big(\int_G\frac{\mathcal{A}^2}{P}\,\mathrm{d}x\Big)^{1/2}\\
&\leq -2(d_a-\epsilon)\int_G\frac{\mathcal{A}^2}{P}\,\mathrm{d}x+\frac{K^2C_{\mathbf{B}_m}^2|G|}{2\epsilon\,\underline{P}}\,e^{-2d_m t}.
\end{aligned} \tag{4.5}
$$

In the third line we used Youngs's inequality to split the second term and introduced the azimuthal decay rate $d_a:=\eta\mu_1\,\underline{P}/\overline{P}$. Applying Gronwall's inequality on (4.5) yields then the exponential decay of the quantity $\int_G\mathcal{A}^2/P\,\mathrm{d}x$:

$$
\int_G\frac{\mathcal{A}(\mathbf{x},t)}{P(\mathbf{x},t)}\,\mathrm{d}x\leq\int_G\frac{\mathcal{A}_0^2(\mathbf{x})}{P_0(\mathbf{x})}\,\mathrm{d}x\ e^{-2(d_a-\epsilon)t}+\frac{K^2C_{\mathbf{B}_m}^2|G|}{2\epsilon\,\underline{P}}\,\frac{e^{-2(d_a-\epsilon)t}-e^{-2d_m t}}{2(d_m-d_a+\epsilon)}\,,\quad t\geq 0.
$$

Eliminating, finally, the auxiliary function P, we end up with the energy estimate

$$
\begin{aligned}
\int_G\mathcal{A}^2(\mathbf{x},t)\,\mathrm{d}x\leq\overline{P}\int_G\frac{\mathcal{A}^2(\mathbf{x},t)}{P(\mathbf{x},t)}\,\mathrm{d}x\leq\frac{\overline{P}}{\underline{P}_0}\int_G\mathcal{A}_0^2(\mathbf{x})\,\mathrm{d}x\ e^{-2(d_a-\epsilon)t}&\\
+\frac{\overline{P}}{\underline{P}}\,\frac{K^2C_{\mathbf{B}_m}^2|G|}{4\epsilon|d_m-d_a+\epsilon|}\ e^{-2\min\{d_a-\epsilon,\,d_m\}t}=:C_A\,e^{-2\min\{d_a-\epsilon,\,d_m\}t},&
\end{aligned} \tag{4.6}
$$

where C_A depends on the initial value $\mathcal{A}_0$, the fraction $\overline{P}/\underline{P}$, K, $C_{\mathbf{B}_m}$, $|G|$, $d_m - d_a$ and $\epsilon > 0$ (chosen such that $d_m - d_a + \epsilon \neq 0$).

To obtain bounds on $\mathbf{B}_a$ and $\mathbf{J}_m = \nabla \times \mathbf{B}_a$ in the maximum norm we use once more Theorem 1, which holds literally also in the case that the 'boundary condition' $(3.2)_{2,3,4}$ is replaced by a Dirichlet condition (Kaiser 2012b). Resolving the divergence form, the governing equation $(2.17)_1$ of the azimuthal problem in $\mathbb{R}^5$ takes the form of $(3.2)_1$. So, Theorem 1 may be applied to solutions of (2.17). Choosing $n = 5$ and $k = 3$, the main conditions (3.4) and (3.5) are satisfied by (3.9) with $d_f = d = d_m$ and (4.6) with $d_0 = \min\{d_a - \epsilon,\, d_m\}$, respectively. Note that the intrinsic decay rate d_i of (2.17) coincides with the azimuthal free decay rate (see section 6), thus $d_a \leq d_i$. By (3.10) we obtain then the bound

$$\sup_{\overline{G}_5} |\mathcal{A}(\cdot,t)| + \sup_{\overline{G}_5} |\nabla \mathcal{A}(\cdot,t)| \leq \widetilde{C}\, e^{-\min\{d_a-\epsilon,\, d_m\}t}, \qquad t \geq 0$$

and hence for all time

$$\sup_{\overline{G}} \left| \frac{1}{\rho}\, \mathbf{B}_a(\cdot,t) \right| = \sup_{\overline{G}_5} |\mathcal{A}(\cdot,t)| \leq \widetilde{C}_{\mathbf{B}_a}\, e^{-\min\{d_a-\epsilon,\, d_m\}t} \tag{4.7}$$

and

$$\sup_{\overline{G}} |\mathbf{J}_m(\cdot,t)| \leq 2 \sup_{\overline{G}_5} |\mathcal{A}(\cdot,t)| + R \sup_{\overline{G}_5} |\nabla \mathcal{A}(\cdot,t)| \leq \widetilde{C}_{\mathbf{J}_m}\, e^{-\min\{d_a-\epsilon,\, d_m\}t}. \tag{4.8}$$

The constants $\widetilde{C}_{\mathbf{B}_a}$ and $\widetilde{C}_{\mathbf{J}_m}$ depend, as in the meridional case, on bounds on the derivatives of the data η and $\mathbf{v}$ up to fourth order; moreover on η_0, the conducting region G, the initial values $\mathcal{M}_0$ and $\mathcal{A}_0$, and the 'balance parameter' ϵ. All other dependencies can be eliminated by substitution; for instance the 'oscillation' $\overline{P}/\underline{P}$ of the auxiliary function P depends, according to Theorem 1, on no other parameters when the initial value $P_0 = const = 1$ is chosen.

5 Meridional decay rates

In this section we present the example of a flow field, for which the meridional decay rate shrinks exponentially fast to zero with respect to the amplitude of the flow. For simpler flows, viz. purely radial or purely non-radial flows we prove the opposite behaviour, i.e. even for large amplitudes the meridional decay rates do not drop significantly below the corresponding free decay rates. These results are obtained by constructing suitable sub- and supersolutions for the dynamo problem. Their significance depends on the following maximum principle that is formulated according to our needs.

Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂G and complement $\widehat{G}$. Let, furthermore, G be decomposed into a finite number of subdomains $G_1, \ldots, G_k$ by smooth hypersurfaces $\Gamma_1, \ldots, \Gamma_l$ such that $G = \bigcup_{i=1}^k \overline{G}_i$ and $G_i \cap G_j = \emptyset$ for $i \neq j$. So, ∂G_i may have parts in $\Gamma := \bigcup_{j=1}^l \Gamma_j$ and ∂G; it is smooth up to possible intersections of the bounding components Γ_j and ∂G. To take care of a possible symmetry axis S in the

x_n-direction we use the notation $\mathbf{x} = (\boldsymbol{\rho}, x_n)$ with $\rho = |\boldsymbol{\rho}| = \left(\sum_{j=1}^{n-1} x_j^2\right)^{1/2}$. L denotes the elliptic operator $L := a\Delta + \mathbf{b} \cdot \nabla + c$ with coefficients a, $\mathbf{b}$, c satisfying the regularity requirements a, ∇a, $\rho\mathbf{b}$, $\rho c \in \bigcap_{i=1}^k C^0(\overline{G}_i \times [0,T]) \cap C^0(\widehat{G} \times [0,T])$ and $a \in C^0(\overline{G} \times [0,T])$ with $a \geq a_0 > 0$. So, $\mathbf{b}$ and c are possibly unbounded at S and need not be continuous over Γ, whereas a is supposed to be continuous in all G. Functions $u = u(\boldsymbol{\rho}, z, t)$ then satisfy the system of inequalities

$$\left.\begin{array}{lll} \partial_t u \leq Lu & \text{in } G_i \times (0,T) & \text{for } i = 1, \ldots, k, \\ 0 \leq Lu & \text{in } \widehat{G} \times (0,T), & \\ u \text{ and } \nabla u \text{ continuous} & \text{in } \mathbb{R}^n \times (0,T), & \\ u(\boldsymbol{\rho}, \cdot, \cdot) \leq 0 & \text{for } \rho \to 0, & \\ u(\boldsymbol{\rho}, z, \cdot) \leq 0 & \text{for } |\mathbf{x}| \to \infty, & \\ u(\cdot, \cdot, 0) = u_0 & \text{on } G \times \{t = 0\}. & \end{array}\right\} \tag{5.1}$$

in a classical sense if all derivatives exist as continuous functions, i.e. $u \in \bigcap_{i=1}^k C_1^2(G_i \times (0,T)) \cap C^2(\widehat{G} \times (0,T))$. For such functions we have the following result

Theorem 3 (Maximum principle) *For functions $u \in (\overline{G} \times [0,T])$ with $T > 0$, which satisfy (5.1) with $c \leq 0$ in a classical sense, holds*

$$\sup_{\mathbf{R}^n \times (0,T]} u \leq \max_{\overline{G}} u_0^+ .$$

Here, u_0^+ means the positive part of u_0, i.e. $u_0^+(\mathbf{x}) := \max\{u_0(\mathbf{x}), 0\}$, and $u \in C(\overline{G} \times [0,T])$ implies that u takes continuously its initial value u_0 on G. A proof of Theorem 3 is given in Appendix B; we make here only a few comments: our result differs from similar ones in (Backus 1957) or (Ivers & James 1984) in that several separate conducting regions are admitted. This allows us to consider discontinuous flows, which typically have no classical solutions in all G. Theorem 3 states, in case of a positive initial maximum, that u can never exceed this value and, in case of a negative initial maximum, that u can never exceed zero.[4] Theorem 3 holds likewise in the bounded case, where $(5.1)_{2,5}$ are replaced by the boundary condition $u|_{\partial G \times (0,T)} \leq 0$.

In this section it is convenient to use polar coordinates (r, θ) in the meridional plane in the form $\rho = r \sin\theta$, $z = r\cos\theta$ with $r > 0$ and $0 < \theta < \pi$. The symmetry axis is then represented by $\theta = 0$ and π, $r > 0$. Equation (2.5) for the meridional scalar $M = M(r, \theta, t)$ takes then the form

$$\partial_t M - \eta\Big(\partial_r^2 M + \frac{\sin\theta}{r^2}\, \partial_\theta\Big(\frac{1}{\sin\theta}\, \partial_\theta M\Big)\Big) + v_r\, \partial_r M + \frac{1}{r}\, v_\theta\, \partial_\theta M = 0 \tag{5.2}$$

[4] We take the opportunity to correct the maximum principle in (Kaiser 2007), where we missed the case of a negative initial maximum: in Lemma 1 q_0 should be replaced by q_0^+. All consequences we have drawn there from Lemma 1 remain untouched by this correction.

and the axis condition (2.19) reads

$$M(\cdot,\theta,\cdot) = O(\sin^2\theta) \qquad \text{for } \theta \to 0 \text{ or } \pi. \tag{5.3}$$

Obviously, (r,θ,ϕ) constitute spherical coordinates in $\mathbb{R}^3$.

5.1 Exemplary slow decay

Let us consider the piecewise constant (discontinuous) velocity field $\mathbf{v}$ (in spherical coordinates) in $B_1 \subset \mathbb{R}^3$:

$$\left.\begin{aligned} v_r &= \begin{cases} -3\,c & \text{for } 1/3 < r < 2/3 \\ 3\,c & \text{for } 2/3 < r < 1 \end{cases}, \quad 0 < \theta < \pi, \\ \frac{v_\theta}{r\sin\theta} &= \begin{cases} -9\,c & \text{for } 0 < \theta < \pi/2 \\ 9\,c & \text{for } \pi/2 < \theta < \pi \end{cases}, \quad 1/3 < r < 1, \end{aligned} \quad t \geq 0 \quad \right\} \tag{5.4}$$

with some constant $c > 0$; otherwise $\mathbf{v}$ is zero, in particular, $\mathbf{v} = 0$ in $B_{1/3}$ and $v_\phi \equiv 0$. Let M be a solution of the meridional problem (2.14) with this flow field, with $\eta = 1$ and with initial value $M_0 \geq 0$. In general, M will not be a classical solution in $G = B_1$, but M as well as $-M$ will satisfy the premises of Theorem 3, especially (5.1) with B_1 subdivided into $B_{1/3}$ and four spherical half-shells. Applying Theorem 3 on $-M$ then yields

$$\inf_{\mathbb{R}^3\times(0,T]} M \;\geq\; \min_{\mathbf{x}\in\overline{B}_1}\{\min\{M_0(\mathbf{x}),0\}\} = 0 \tag{5.5}$$

for any $T > 0$.

In the following we construct a *subsolution* $\underline{M}$ of (5.2) (i.e. the left-hand side in (5.2) is non-positive) with decay rate d_m on the half-annulus $1/3 < r < 1$, $0 < \theta < \pi$ satisfying (5.3) and the boundary condition $\underline{M}|_{r=1/3} = \underline{M}|_{r=1} = 0$. In $\mathbb{R}^3$ this region corresponds to the spherical shell $SS := B_1 \setminus \overline{B}_{1/3}$. The function $\underline{M} - M$ then satisfies on SS the bounded version of (5.1), in particular, we have by (5.5), $(\underline{M} - M)|_{\partial SS} \leq 0$. So, choosing $\underline{M}_0 \leq M_0$, Theorem 3 yields $\sup_{SS\times(0,T]}(\underline{M} - M) \leq 0$ for any $T > 0$ or, equivalently, $M(\cdot,t) \geq \underline{M}(\cdot,t)$ for any $t \geq 0$, i.e. d_m is an upper bound on the decay rate of M.

To obtain a suitable subsolution $\underline{M}$ we make the ansatz

$$\underline{M}(r,\theta,t) := f(r)\,g(\theta)\,e^{-d_m t} \tag{5.6}$$

with non-negative functions f and g, and a constant $d_m > 0$. The construction of f and g is based on the auxiliary function

$$h(s) := e^{-c\,s/2}\sinh\sqrt{(c/2)^2 - d}\,s, \qquad 0 < s < 1, \tag{5.7}$$

where the parameters c and d are related by

$$\sqrt{(c/2)^2 - d}\,\coth\sqrt{(c/2)^2 - d} = c/2,$$

which defines a function $d = D(c)$ that behaves asymptotically like

$$d = D(c) \sim c^2 e^{-c} \quad \text{for } c \to \infty. \tag{5.8}$$

h is the unique solution of the boundary value problem

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\, h + c\,\frac{\mathrm{d}}{\mathrm{d}s}\, h + d\, h = 0, \qquad h(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}s}\, h(1) = 0. \tag{5.9}$$

Thus, with the substitution $t := 1 - s$, h satisfies the inequality

$$(1 - t^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, h - c(1 - t^2)\frac{\mathrm{d}}{\mathrm{d}t}\, h + d\, h \geq 0, \qquad 0 < t < 1,$$

or, after the further substitution $t := \cos\theta$,

$$\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\frac{1}{\sin\theta}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\, g_1\Big) + c\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\, g_1 + d\, g_1 \geq 0, \qquad 0 < \theta < \frac{\pi}{2},$$

where $g_1(\theta) := h(t)$. At the boundaries we have $g_1(0) = \mathrm{d}/\mathrm{d}\theta\, g_1(\pi/2) = 0$. Introducing v_θ from (5.4) and observing that $\mathrm{d}/\mathrm{d}\theta\, g_1 \geq 0$ we have, finally,

$$\sin\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\frac{1}{\sin\theta}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\, g_1\Big) - r v_\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta}\, g_1 + d\, g_1 \geq 0, \qquad 0 < \theta < \frac{\pi}{2}, \quad \frac{1}{3} < r < 1. \tag{5.10}$$

By reflection at $\theta = \pi/2$ we find that $g_2(\theta) := g_1(\pi - \theta)$ satisfies inequality (5.10) in the region $\pi/2 < \theta < \pi$, $1/3 < r < 1$.

Concerning the radial part we start from (5.9) with d replaced by $\widetilde{d} := d_m/9 - d$ and find by the substitution $r := (1 + s)/3$ for $f_1(r) := h(s)$ the inequality

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\, f_1 - r^2 v_r\,\frac{\mathrm{d}}{\mathrm{d}r}\, f_1 + r^2 d_m f_1 - d\, f_1 \geq 0, \qquad 1/3 < r < 2/3 \tag{5.11}$$

and the boundary conditions $f_1(1/3) = \mathrm{d}/\mathrm{d}r\, f_1(2/3) = 0$. An analogous calculation yields $f_2(r)$ satisfying (5.11) on $2/3 < r < 1$ with boundary conditions $\mathrm{d}/\mathrm{d}r\, f_2(2/3) = f_2(1) = 0$.

The functions $f_{1,2}$ and $g_{1,2}$ constitute now the subsolution (5.6) on their respective domains: (5.10) and (5.11) imply (5.2) (as inequality) with $\eta \equiv 1$ and one easily checks C^1-smoothness over the subdomains, the asymptotic condition (5.3) and the boundary conditions $\underline{M}(1/3, \cdot, \cdot) = \underline{M}(2/3, \cdot, \cdot) = 0$. For large flow amplitudes c, the quantities d and $\widetilde{d}$ and hence d_m obey (5.8), which manifests the (exponentially) slow decay of M.

5.2 Purely radial and non-radial flows

A *supersolution* $\overline{M}$ (i.e. $-\overline{M}$ satisfies (5.1)) with decay rate d_m that bounds a solution M at $t = 0$ does so for all time, hence d_m provides a lower bound on the decay rate of M. This follows by Theorem 3 applied on $M - \overline{M}$ and on $-M - \overline{M}$.

We consider first purely radial flows, i.e. $v_\theta \equiv 0$, in an arbitrary conducting region G. Concerning v_r the following bounds will play a role:

$$\left.\begin{array}{rll} -v_r \leq 2\,c_1\,\eta_0/R & \text{for } 0 < r < R/2, & \\ & & 0<\theta<\pi, \quad t \geq 0, \\ v_r \leq 2\,c_2\,\eta_0/R & \text{for } R/2 < r < R, & \end{array}\right\} \tag{5.12}$$

where η_0 is a lower bound on η and R the radius of a ball enclosing G. For large flow amplitudes c_1, c_2 a supersolution is then given by

$$\overline{M}_r(r,\theta,t) := f(r)\,\sin^2(\theta)\,e^{-d_m^r t}, \tag{5.13}$$

where $d_m^r := 8\eta_0/R^2$ and

$$f(x) := \begin{cases} a_1\sqrt{x}\,e^{-c_1 x} I_{3/2}(b_1 x) & 0 < x < 1/2, \\ a_2\sqrt{x}\,e^{c_2 x}(K_{3/2}(b_2 x) + a_3 I_{3/2}(b_2 x)) & 1/2 < x < 1, \\ x^{-1} & x > 1 \end{cases}$$

with $x := r/R$ and $b_i := (c_i^2 - d_i)^{1/2}$ $(i = 1, 2)$. I_ν and K_ν denote modified Bessel functions of order ν and I'_ν and K'_ν their derivatives. Thus $f(x)$ is a solution of

$$x^2 f'' - 2(-1)^i c_i\, x^2 f' + d_i\, x^2 f - 2f = 0 \qquad (i = 1, 2) \tag{5.14}$$

on $0 < x < 1/2$ $(i = 1)$ and $1/2 < x < 1$ $(i = 2)$, respectively. To ensure C^1-smoothness we require $f'(1/2-) = 0$ and $f'(1/2+) = 0$, which yield

$$(1 - c_1) I_{3/2}(b_1/2) + b_1 I'_{3/2}(b_1/2) = 0 \tag{5.15}$$

and

$$(1 + c_2)(K_{3/2}(b_2/2) + a_3 I_{3/2}(b_2/2)) + b_2(K'_{3/2}(b_2/2) + a_3 I'_{3/2}(b_2/2)) = 0, \tag{5.16}$$

respectively. a_3 is determined by the condition $f'(1-) = -f(1-)$ and a_1 and a_2 are chosen such that f is continous. Relation (5.15) determines a function $d_1 = D_1(c_1)$ that for large arguments decreases monotonically to the asymptotic value $\lim_{c_1\to\infty} D_1(c_1) = 8$.[5] Similarly, (5.16) determines a function D_2 with the same asymptotic property. Note that f is monotonically increasing for $0 < x < 1/2$ and decreasing for $1/2 < x < 1$, which allows us to use the bounds (5.12). Thus, $\overline{M}_r$ satisfies (5.2) (as inequality) in B_R and hence in G. The remaining conditions in (5.1) are easily checked. This verifies $\overline{M}_r$ to be a supersolution with decay rate $8\eta_0/R^2$, which differs from the free decay rate $\pi^2\eta_0/R^2$ by less than 20%. This bound cannot be improved since for a piecewise constant flow field in a ball according to (5.12) with c_1/c_2 chosen such that $d_1 = d_2$, (5.13) becomes an exact solution of the dynamo problem.

[5] This analysis has to be done with some care but it is elementary since $I_{3/2}$ and $K_{3/2}$ can be expressed by elementary functions (cf. Abramowitz & Stegun 1972, p. 443).

For purely non-radial flows holds $v_r \equiv 0$. With the bounds on v_θ,

$$\begin{aligned} -rv_\theta &\le c\,\sin\theta\,\eta_0 && \text{for } 0<\theta<\pi/2, \\ rv_\theta &\le c\,\sin\theta\,\eta_0 && \text{for } \pi/2<\theta<\pi, \end{aligned} \qquad r<R, \quad t\ge 0,$$

a supersolution reads now:

$$\overline{M}_{nr}(r,\theta,t) := f(r)\,g(\theta)\,e^{-d_m^{nr}t},$$

where $d_m^{nr} := \pi^2\eta_0/(4R^2)$, $f(r) := \sin\pi r/(2R)$ for $r<R$ and $f(r) := 1$ for $r>R$, and $g(\theta) := h(1-\cos\theta)$ for $0<\theta<\pi/2$ and $g(\theta) := h(1+\cos\theta)$ for $\pi/2<\theta<\pi$ with h being the function (5.7). The verification of the conditions (5.1) is left to the reader.

6 Incompressible fluids of constant conductivity

When the flow field $\mathbf{v}$ is such that $\nabla\cdot\mathbf{v} = 0$ and the diffusivity η is constant, much larger decay rates can be obtained than without these restrictions. In fact, the decay rates of the meridional and the azimuthal field are those of axisymmetric free decay. If the conducting region is a ball, explicit calculations show that these decay rates coincide with the well-known poloidal and toroidal free decay rates, respectively. These numbers provide, moreover, lower decay rate bounds for any region enclosed by these balls. The basis of these results is a careful evaluation of the corresponding energy balances and of the associated variational problems. We start with the meridional case.

Multiplying $(2.14)_1$ by M, integrating over the regularized domain G_ϵ introduced in section 4, inegrating by parts and using the boundary condition $\mathbf{n}\cdot\mathbf{v} = 0$ on S_ϵ yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{G_\epsilon} M^2\mathrm{d}x &= \int_{G_\epsilon}\Big(\eta\Delta M - 2\eta\,\frac{\nabla\rho}{\rho}\cdot\nabla M - \mathbf{v}\cdot\nabla M\Big)M\,\mathrm{d}x \\ &= -\eta\int_{G_\epsilon}|\nabla M|^2\mathrm{d}x + \eta\int_{S_\epsilon} M\,\mathbf{n}\cdot\nabla M\,\mathrm{d}s - \eta\int_{S_\epsilon}\frac{\mathbf{n}\cdot\nabla\rho}{\rho}\,M^2\mathrm{d}s \\ &\quad -\eta\int_{C_\epsilon} M\,\nabla\rho\cdot\nabla M\,\mathrm{d}s + \eta\int_{C_\epsilon}\frac{M^2}{\rho}\,\mathrm{d}s + \frac{1}{2}\int_{C_\epsilon}\nabla\rho\cdot\mathbf{v}\,M^2\mathrm{d}s. \end{aligned} \tag{6.1}$$

An analogous calculation yields in $\widehat{G}_\epsilon := \{\mathbf{x}\in\widehat{G} : \rho(\mathbf{x}) > \epsilon\}$:

$$\begin{aligned} 0 = \eta\int_{\widehat{G}_\epsilon}\Big(\Delta M - 2\,\frac{\nabla\rho}{\rho}\cdot\nabla M\Big)M\,\mathrm{d}x &= -\eta\int_{\widehat{G}_\epsilon}|\nabla M|^2\mathrm{d}x - \eta\int_{S_\epsilon} M\,\mathbf{n}\cdot\nabla M\,\mathrm{d}s \\ +\eta\int_{S_\epsilon}\frac{\mathbf{n}\cdot\nabla\rho}{\rho}\,M^2\mathrm{d}s &- \eta\int_{\widehat{C}_\epsilon} M\,\nabla\rho\cdot\nabla M\,\mathrm{d}s + \eta\int_{\widehat{C}_\epsilon}\frac{M^2}{\rho}\,\mathrm{d}s, \end{aligned} \tag{6.2}$$

where we used the asymptotic conditions $(2.14)_5$ and (2.18) and the notation $\widehat{C}_\epsilon := \{\mathbf{x}\in\widehat{G} : \rho(\mathbf{x}) = \epsilon\}$. Summing up (6.1) and (6.2) and letting $\epsilon\to 0$ while observing $(2.14)_{3,4}$

yields, finally, the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G M^2\mathrm{d}x = -2\eta\int_{\mathbb{R}^3}|\nabla M|^2\mathrm{d}x.$$

So, solving the variational problem

$$\inf_{0\neq M\in S_m}\frac{\int_{\mathbb{R}^3}|\nabla M|^2\mathrm{d}x}{\int_G M^2\mathrm{d}x} =: \lambda_1, \tag{6.3}$$

where S_m means the set of axisymmetric functions that satisfy $(2.14)_{3,4,5}$, yields exponential decay of the energy with rate $2d_m^0 := 2\eta\lambda_1$. The Euler-Lagrange equations associated to (6.3) are obtained by a calculation analogous to that of (6.1), (6.2). They amount to the eigenvalue problem

$$\left.\begin{array}{cl} -\Delta M + 2\,\dfrac{\nabla\rho}{\rho}\cdot\nabla M = \lambda\, M & \text{in } G, \\ \Delta M - 2\,\dfrac{\nabla\rho}{\rho}\cdot\nabla M = 0 & \text{in } \widehat{G}, \\ M \text{ and } \nabla M \text{ continuous} & \text{in } \mathbb{R}^3, \\ M(\mathbf{x})\to 0\ ,\ \nabla M(\mathbf{x}) = O(\rho) & \text{for } \rho\to 0, \\ \dfrac{1}{\rho}\,\nabla M(\mathbf{x})\to 0 & \text{for } |\mathbf{x}|\to\infty, \end{array}\right\} \tag{6.4}$$

whose lowest eigenvalue λ_1 solves (6.3) (see Appendix C). Obviously, (6.4) and hence λ_1 still depend on G but not on $\mathbf{v}$; thus, λ_1 defines the free decay rate in G.

When reformulated in $\mathbb{R}^5$, (6.4) represents an eigenvalue problem for the Laplacian. This eigenvalue problem has explicitly been solved for the case that $G\subset\mathbb{R}^n$ is a ball B_R of radius R (see Appendix C). For $n=5$ we find $\lambda_1 = (\pi/R)^2$, hence $d_m^0 = \eta\pi^2/R^2$, which coincides with the poloidal free decay rate.

Note that Backus considers in the ref. (Backus 1957) the more general variational problem (28), which for $\rho=\kappa=1$ coincides with our problem (6.3). Neglecting axisymmetry, in particular, the axis-condition $(6.4)_4$, Backus derives the Euler-Lagrange equations (30) and (31), which formally coincide with those describing poloidal free-decay modes in $\mathbb{R}^3$. Since in the axisymmetric situation the asymptotic condition at infinity is weaker and no zero-mean condition applies, he finds in balls B_R the lowest eigenvalue $(\pi/2R)^2$, which suggests a decay time that is 4 times larger. This result, however, does not seem to be correct. In fact, the corresponding eigenfunction is spherically symmetric; so, its gradient does not vanish on the symmetry axis and, therefore, is not admissible in the representation (2.2) of axisymmetric fields.

Concerning the azimuthal case we can make use of the calculations already performed in section 4. With the notation introduced there the energy balance can be read off (4.4) by setting $P\equiv 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G \mathcal{A}^2\mathrm{d}x = -2\eta\int_G|\nabla\mathcal{A}|^2\mathrm{d}x - 4\pi\eta\int_L \mathcal{A}^2\mathrm{d}z + 2\int_G \mathcal{A}\,f\,\mathrm{d}x. \tag{6.5}$$

Neglecting the last term, the azimuthal variational problem is then

$$\inf_{0\neq\mathcal{A}\in S_a} \frac{\int_G |\nabla\mathcal{A}|^2 \mathrm{d}x + 2\pi \int_L \mathcal{A}^2 \mathrm{d}z}{\int_G \mathcal{A}^2 \mathrm{d}x} =: \mu_1, \tag{6.6}$$

where S_a means the set of axisymmetric differentiable functions vanishing on ∂G. The associated Euler-Lagrange equations read

$$\left.\begin{aligned} -\nabla\cdot\Big(\frac{1}{\rho^2}\,\nabla(\rho^2\mathcal{A})\Big) = \mu\,\mathcal{A} &\quad \text{in } G,\\ \mathcal{A} = 0 &\quad \text{on } \partial G, \end{aligned}\right\} \tag{6.7}$$

where the lowest eigenvalue μ_1 solves the variational problem (6.6). This is again more transparent when reformulated in $\mathbb{R}^5$. Problem (6.7) becomes then the well-known eigenvalue problem for the Laplacian with Dirichlet boundary condition:

$$\left.\begin{aligned} -\Delta\mathcal{A} = \mu\,\mathcal{A} &\quad \text{in } G_5,\\ \mathcal{A} = 0 &\quad \text{on } \partial G_5. \end{aligned}\right\} \tag{6.8}$$

In balls B_R the eigenvalues are explicitly known; in particular, there is $\mu_1 = (i_1^1/R)^2$, where i_1^1 is the lowest positive zero of the first spherical Bessel function j_1 (cf. Folland 1995, p. 106).

For the azimuthal field energy, the decay estimate (4.6) holds then with $P \equiv P_0 = \overline{P} = \underline{P} = 1$ and azimuthal decay rate $d_a^0 := \eta\mu_1$. In balls we have $d_a^0 = \eta(i_1^1/R)^2$, which coincides with the toroidal free decay rate.[6] Note that (4.6) contains the possibility that in the presence of a meridional field, the azimuthal decay is also dominated by the (lower) meridional decay rate.

Pointwise decay can again be obtained by means of Theorem 1. In the azimuthal case without meridional field pointwise decay holds with almost the free decay rate d_a^0 since the five-dimensional energy (which is required by Theorem 1) is dominated by the three-dimensional energy. This is not so in the meridional case. Only when using additional information, five-dimensional energy decay (with a lowered decay rate) can be established: splitting the conducting region G into G_ϵ and its complement and using (3.1) we can estimate

$$\begin{aligned} \int_{G_5} \mathcal{M}(\mathbf{x},t)^2 \mathrm{d}x &= \frac{|S_3|}{2\pi} \int_G \Big(\frac{1}{\rho^2} M(\mathbf{x},t)\Big)^2 \rho^2 \mathrm{d}x \\ &\le \frac{|S_3|}{2\pi}\,\frac{1}{\epsilon^2} \int_{G_\epsilon} M(\mathbf{x},t)^2 \mathrm{d}x + C^2 \int_{G\setminus G_\epsilon} g(\rho)^2 \frac{1}{\rho^2}\,\mathrm{d}x \\ &\le \frac{\widetilde{C}}{2\epsilon^2}\, e^{-2d_m^0 t} + \frac{\widetilde{C}}{2}\,\epsilon^2. \end{aligned}$$

[6]This confirms again the assertion of Braginskii (1965), although he seems not to have considered the correct variational problem. Backus (1957) finds with ingenious but not completely rigorous arguments again a lower decay rate, viz. $\eta(\pi/R)^2$.

Setting $\epsilon := \exp(-d_m^0 t/2)$ we thus have

$$\int_{G_5} \mathcal{M}(\mathbf{x},t)^2 \mathrm{d}x \le \widetilde{C}\, e^{-d_m^0 t}$$

and hence by (3.12) pointwise decay of the meridional field with (at least) half the free decay rate (which is supposedly not optimal).

Finally, one should note that eigenvalues of (6.8) are monotonous with respect to the domain, i.e. $G \subset \widetilde{G}$ implies $\mu \ge \widetilde{\mu}$, where μ and $\widetilde{\mu}$ belong to G and $\widetilde{G}$, respectively (cf. Courant & Hilbert 1961, p. 409). The eigenvalues λ of (6.4) share this property (see Appendix C). So, the decay rates d_m^0 and d_a^0 in arbitrary (axisymmetric) domains G can always be estimated by those for balls enclosing G.

Appendix A

This appendix expounds the necessary changes in the proof of (Lortz et al. 1984, Theorem 2) to be applicable to Theorem 2. First we rewrite system (4.1) in $\mathbb{R}^5$. In view of (2.17) one obtains

$$\left.\begin{aligned} \partial_t P - \nabla\cdot(\eta\nabla P) + \nabla\cdot(\mathbf{v}_m^{(5)} P) + c\,P = 0 \quad & \text{in } G_5\times(0,T),\\ \mathbf{n}\cdot\nabla P = 0 \quad & \text{on } \partial G_5\times(0,T),\\ P(\cdot\,,0) = P_0 > 0 \quad & \text{on } G_5\times\{t=0\} \end{aligned}\right\} \tag{A.1}$$

with $c := -2(\partial_\rho\eta/\rho + v_\rho/\rho)$ and arbitrary $T>0$. System (A.1) differs from system (2.2) in (Lortz et al. 1984)[7] by the zeroth-order term $c\,P$. Classical solutions are guaranteed for both systems by the sufficient conditions $\partial G \in C^3$, η and $\mathbf{v}_m^{(5)} \in C_1^2(\overline{G}_5\times[0,T])$ with $\mathbf{n}\cdot\mathbf{v}_m^{(5)}|_{\partial G\times[0,T]} = 0$, $c \in C_1^1(\overline{G}_5\times[0,T])$ and $P_0 \in C^3(\overline{G}_5)$ with $\mathbf{n}\cdot\nabla P_0|_{\partial G} = 0$, which imply the precise Hölder conditions formulated, e.g., in (Ladyzenskaja et al. 1968, chap. IV, Theorem 5.3). The proof of (Lortz et al. 1984, Theorem 2) refers to such classical solutions.

The governing equation (LMS2.2a) enters the proof two times, viz. in the derivations of the inequalities (LMS3.1) and (LMS3.12). We show in the following that the zeroth-order term in $(\mathrm{A.1})_1$ does not invalidate these inequalities, it merely modifies the bounds that depend then, additionally, on c.

Modified proof of (LMS3.1): Multiplying $(\mathrm{A.1})_1$ by $P^{\gamma-1}$ with $\gamma \ne 0,1$, integrating over G, integrating by parts and using the boundary conditions for P and $\mathbf{v}_m^{(5)}$ yields

$$\begin{aligned} \frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_G P^\gamma\,\mathrm{d}x &= \int_G \partial_t P\, P^\gamma\,\mathrm{d}x = \int_G \big\{\nabla\cdot(\eta\nabla P - \mathbf{v}_m^{(5)}P)P^{\gamma-1} - c\,P^\gamma\big\}\mathrm{d}x \\ &= \int_G \big\{(1-\gamma)(\eta P^{\gamma-2}|\nabla P|^2 - P^{\gamma-1}\mathbf{v}_m^{(5)}\cdot\nabla P) - c\,P^\gamma\big\}\mathrm{d}x. \end{aligned} \tag{A.2}$$

[7] Equation numbers referring to (Lortz et al. 1984) are henceforth marked by the suffix 'LMS', thus, '(LMS2.2)'.

With the bounds $\eta \geq \eta_0$, $|\mathbf{v}_m^{(5)}| \leq K$ and $|c| \leq K$, and by Young's inequality, (A.2) can be estimated as follows:

$$\begin{aligned}\frac{1}{\gamma(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_G P^\gamma\,\mathrm{d}x \;&\leq -\eta_0\int_G P^{\gamma-2}|\nabla P|^2\mathrm{d}x + K\int_G P^{\gamma-1}|\nabla P|\,\mathrm{d}x + \frac{K}{|\gamma-1|}\int_G P^\gamma\mathrm{d}x\\ &\leq -\frac{3}{4}\eta_0\int_G P^{\gamma-2}|\nabla P|^2\mathrm{d}x + \Big(\frac{K^2}{\eta_0}+\frac{K}{|\gamma-1|}\Big)\int_G P^\gamma\mathrm{d}x.\end{aligned}$$

Finally, multiplying by $\phi(t)$, integrating over $[t_1,t_2]$ and integrating by parts yields (LMS3.1) with the constant K^2/η_0 replaced by $K^2/\eta_0 + K/|\gamma-1|$. In the course of proof the parameter γ takes infinitely many values; however, always holds $|\gamma-1|^{-1} \leq n+1$ if $n \geq 3$, in particular, $|\gamma-1|^{-1} \leq 6$ in $\mathbb{R}^5$ (Lortz et al. 1984, p. 688 and p. 690). Thus, (LMS3.1) holds with an enlarged constant.

Modified proof of (LMS3.12): Multiplying $(\mathrm{A}.1)_1$ by P^{-1} and integrating over G yields analogously:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_G \log P\,\mathrm{d}x \;&= \int_G\big\{\nabla\cdot(\eta\nabla P - \mathbf{v}_m^{(5)}P)P^{-1} - c\big\}\mathrm{d}x\\ &= \int_G \eta|\nabla\log P|^2\mathrm{d}x - \int_G \nabla(\log P)\cdot\mathbf{v}_m^{(5)}\,\mathrm{d}x - \int_G c\,\mathrm{d}x\\ &\geq \frac{1}{2}\eta_0\int_G|\nabla\log P|^2\mathrm{d}x - \frac{1}{2}\frac{K^2}{\eta_0}|G| - K|G|,\end{aligned}$$

which is (LMS3.12) with the constant $K^2|G|/(2\eta_0)$ enlarged by $K|G|$.

Appendix B

Proof of Theorem 3: Let M denote $\sup_{\mathbb{R}^n\times(0,T]} u$, which is finite because of $(5.1)_5$. The case $M \leq 0$ is trivial: by continuity we have $\sup_{\overline{G}} u_0 \leq 0$ and hence $M \leq 0 = \max_{\overline{G}} u_0^+$.

To prove the case $M > 0$ we make use of the elliptic maximum principle and the elliptic boundary derivative theorem for classical solutions as formulated in (Protter & Weinberger 1984, Theorems 6-8 at p. 64ff) and the parabolic maximum principle for weak solutions as formulated in (Lieberman 1996, Theorem 6.25 and Corollary 6.26 at p. 128). We show first that $\sup_{\widehat{G}} u(\cdot,t) < \max_{\overline{G}} u(\cdot,t)$ for any $t \in (0,T]$ and, second, that $\max_{\overline{G}\times[0,T]} u$ is taken at $t=0$, which proves our assertion.

1) Let $t_0 \in (0,T]$ and $\widehat{M}_{t_0} := \sup_{\widehat{G}} u(\cdot,t_0)$. Because of $(5.1)_{4,5}$ we find a ball B with complement $\widehat{B}$ and a solid cylinder SC around the symmetry axis with complement $\widehat{SC}$ such that $u(\cdot,t_0) < \widehat{M}_{t_0}$ in $\widehat{B} \cup SC$. Applying the elliptic maximum principle on the bounded and regularized domain $\widehat{G}\cap B\cap\widehat{SC}$ we find $\widehat{M}_{t_0}$ be attained at some point $(\mathbf{x}_0,t_0) \in \partial\widehat{G} = \partial G$. Furthermore, as $u \neq const$ in $\widehat{G}$ and ∂G being smooth the boundary derivative theorem is applicable with the result

$$\mathbf{n}\cdot\nabla u|_{(\mathbf{x}_0,t_0)} < 0, \tag{B.1}$$

where $\mathbf{n}$ denotes the exterior normal with respect to G at $(\mathbf{x}_0, t_0)$. By $(5.1)_3$ condition (B.1) implies $u(\mathbf{x}, t_0) > \widehat{M}_{t_0}$ for some $\mathbf{x} \in G$ and hence $M > \widehat{M}_{t_0}$ for any t_0, i.e.

$$M > u|_{\partial G \times (0,T]} \, . \tag{B.2}$$

2) As u need not be a classical solution in all $G \times (0, T)$ a parabolic maximum principle for *weak* solutions is better suited to complete te proof. In order to check that u satisfies $(5.1)_1$ in the weak sense of (Lieberman 1996, p. 100 above) we write L in the form

$$L = \nabla \cdot (a\nabla) + (\mathbf{b} - \nabla a) \cdot \nabla + c \tag{B.3}$$

and choose again a solid cylinder SC with boundary C such that $u < M$ in $SC \cap G$, which implies, in particular,

$$M > u|_{(C \cap G)\times(0,T]} \, . \tag{B.4}$$

Let $v \in C^1(\overline{(\widehat{SC} \cap G)} \times [0, T])$ be a non-negative testfunction that vanishes at $\partial(\widehat{SC} \cap G) \times [0, T]$. Integrating vLu over $(\widehat{SC} \cap G_i) \times (0, T)$, integrating the first term by parts and summing over $i = 1, \ldots, k$ reveals that u satisfies indeed $(5.1)_1$ in $(\widehat{SC} \cap G) \times (0, T)$ in the weak sense. Note that the regularity assumptions on u and the coefficients are such that the boundary terms on Γ cancel each other; otherwise, boundedness of the coefficients in (B.3) and the sign condition on c are the only further prerequisites of the parabolic maximum principle. Therefore, M is taken at the 'parabolic boundary', which means by (B.2) and (B.4) that M is taken at $t = 0$.

The bounded version of Theorem 3 is obviously proved by 2) and (B.2), which holds now by virtue of the boundary condition $u|_{\partial G \times (0,T]} \leq 0$.

Appendix C

In this appendix we consider the (not necessarily axisymmetric) eigenvalue problem in $\mathbb{R}^n$ $(n \geq 3)$

$$\left.\begin{array}{rl} -\Delta u = \lambda\, u & \text{in } G, \\ \Delta u = 0 & \text{in } \widehat{G}, \\ u \text{ and } \nabla u \text{ continuous} & \text{in } \mathbb{R}^n, \\ u(\mathbf{x}) \to 0 & \text{for } |\mathbf{x}| \to \infty. \end{array}\right\} \tag{C.1}$$

We give a variational characterization of the eigenvalues, demonstrate their monotonicity with respect to the bounded domain $G \subset \mathbb{R}^n$ and compute them explicitly for balls B_R. The axisymmetric case $n = 5$ is equivalent to the meridional eigenvalue problem (6.4).

For sufficiently smooth boundary the system (C.1) is known to have a countable set of (classical) eigensolutions $\{(u_i, \lambda_i) : i \in \mathbb{N}\}$ with positive real eigenvalues λ_i, which are monotonically increasing with increasing i. The eigenfunctions constitute a complete

orthonormal set in $L^2(G)$, the set of square-integrable functions on G (cf. Kaiser & Uecker 2009).

Concerning the variational character of the eigenvalues observe that (C.1) are just the Euler-Lagrange equations of the variational problem

$$\inf_{0\neq u\in S} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x}{\int_G u^2 \mathrm{d}x} =: \lambda_1 \tag{C.2}$$

with u varying in the set S of functions satisfying $(\mathrm{C.1})_{3,4}$. So, λ_1 is the lowest of the eigenvalues of (C.1) and is realized by the associated eigenfunction u_1. Neglecting questions of degeneracy, the next larger eigenvalue λ_2 then solves (C.2) if S is restricted to the orthogonal complement of u_1. Repeating this procedure any eigenvalue has a variational characterization of type (C.2).

The monotonicity of the eigenvalues with respect to G is now a direct consequence of (C.2), since the denominator is always increasing with increasing G.

If $G = B_R \subset \mathbb{R}^n$, eigenfunctions and eigenvalues can explicitly be expressed in terms of spherical Bessel functions j_ν and n-dimensional spherical harmonics $Y_{\nu\mu}$ (cf. Folland 1995, p. 98ff):

$$\left.\begin{array}{c} u^{(n)}_{l\nu\mu}(\mathbf{x}) := \sqrt{\dfrac{2}{R^n}} \begin{cases} \left(\dfrac{|\mathbf{x}|}{R}\right)^{-k} \dfrac{j_{\nu+k}(i_l^{\nu+k-1}|x|/R)}{j_{\nu+k}(i_l^{\nu+k-1})} Y_{\nu\mu}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|}\right) & \text{in } B_R, \\ \left(\dfrac{|\mathbf{x}|}{R}\right)^{2-n-\nu} Y_{\nu\mu}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|}\right) & \text{in } \widehat{B}_R, \end{cases} \\ \lambda^{(n)}_{l\nu\mu} := (i_l^{\nu+k-1}/R)^2, \end{array}\right\} \tag{C.3}$$

where $k := (n-3)/2$, $l \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, $\mu \in \{1, \ldots, D_\nu\}$ and i_l^ν is the lth positive zero of j_ν. (C.3) generalizes the case $n = 3$ (Kaiser & Uecker 2009, Appendix D) to arbitrary $n \geq 3$. That the set $\{u^{(n)}_{l\nu\mu}\}$ is an orthonormal set of eigenfunctions of (C.1) may be checked by explicit calculations. The completeness follows by analogous considerations as in (Kaiser & Uecker 2009, Appendix D).

For the meridional problem (6.4) the lowest eigenvalue is by $(\mathrm{C.3})_2$ $\lambda^{(5)}_{100} = (i_1^0/R)^2 = (\pi/R)^2$. The corresponding eigenfunction $u^{(5)}_{100}$ is by $(\mathrm{C.3})_1$ spherically symmetric , hence $\rho^2 u^{(5)}_{100}$ is axisymmetric and satisfies the axis condition $(6.4)_4$, and is, therefore, admissible in the meridional problem.

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